Interest rate concepts and fixed income securities
266: Financial Markets and Institutions

Jon Faust

http://e105.org/e266

February 15, 2018

- **Pedagogy note**
  - We introduced a lot of ideas quickly in talking about equity valuation

    Present value, annualized rate, discounted cash flow,. . .
  - I did that quick introduction to make the usage concrete before digging in a bit deeper

    (questionable pedagogy, but it's the call I made)
  - Today, we'll repeat material more slowly/thoroughly, in the context of fixed income securities.
  - There are probably more new terms to learn today than in any other lecture
  - And there are a bunch of fairly simple equations.
  - Definitions and equations are best learned by reading in peace and quiet
  - Thus, today, I'll try to hit the highlights, presuming you'll go study this stuff
- **End of pedagogic rationalizations, start of content.**
- **Interest rates**
  - Financial instruments are inherently intertemporal
  - And thus, making good decisions requires understanding how values change through time.
  - Key concept here is 'interest rate'
  - We used interest rates as discount rates in discussing stock valuation
  - Rates of change through time go by many names depending on the context

    interest rate, discount rate, yield, rate of return, . . .

- Interest rate, yield, and rate of return are often used in an imprecise manner and used differently by different people.

  You'll have to pick this up in context.

▶ **Percent change**

- Q: If child is 5 pounds at birth and grows to 5.5 pounds what is the percent change in weight
- A: $100 \times (\frac{\text{final value}}{\text{initial value}} - 1)$

  Our tyke's weight rose by 10 percent.

▶ **Financial markets**

- I invest, \$5 million today and later it is worth \$5.5 million

  A 10 percent increase.

- A 10 percent increase sounds pretty good, but...

▶ **Rates per unit time**

- I only specified that the second value came *later*.
- If it took 50 years to earn 10 percent, not so good. If it happened overnight, very good.

▶ **Thus,**

- Thus, we usually state our percent changes in rates per unit time.
- In particular, we discuss 'annualized rates of return'

  the return per year.

▶ **Annualizing, warning**

- There are different conventions on how to report annualized rates of return.
- We may get into some details, but for now, I'll say what convention we will use in this course.

▶ **Annualizing in this class**

- Suppose $100 \times X$ is the percent increase over h years, we will use the conventional that the annualized rate, AR, is:

$$(1 + AR) = (1 + X)^{1/h}$$

or

$$AR = (1 + X)^{1/h} - 1$$

- **h can be a fraction**
  - The units on h are years, but h can be a fraction
  - Thus, for a 1-day return: $h = 1/365$.
  - For a one-month return $h = 1/12$.
- **Aside:: Well, actually, ...**
  - If we're being picky, the months differ in length, so each month is not $1/12^{th}$ of a year.
  - For example, January would more precisely be written as $(31/365)^{th}$ of a year
- **Aside:: Well actually,. . .**
  - Most years January is $(31/365)^{th}$ of a year, but in leap years, it is $(31/366)^{th}$
- **Conventions**
  - When I say that there are various conventions about reporting annualized rates, many of these have to do with details like this
  - The supplemental notes have an amusing table about what calendar conventions are used in various financial markets markets.

    e.g., all years have 360 days; all years have 365 days; years have the number of days it says on the calendar.
  - You don't need to know these conventions for this class.
  - But you do need to know that when partial years are involved, different markets have different conventions about annualizing.

    And when large sums are at stake, these conventions may amount to significant sums of money.
- **Repeating the essentials**
- **Three versions of THE formula**
  -

$$
\begin{aligned}
FV &= PV(1+i)^h \\
PV &= \frac{FV}{(1+i)^h} \\
(1+i) &= \left(\frac{FV}{PV}\right)^{1/h}
\end{aligned}
$$

- **Example:**

- I pay \$99.57 for a 3-month Treasury Bill that pays \$100 in 3-months. What is the implied interest rate?
- Using $(1+i) = (FV/PV)^{1/h}$,

$$(1+i) = \left(\frac{100}{99.57}\right)^{\frac{1}{3/12}}$$

my holding period is $h = (3/12)^{th}$s of a year.

- $i \approx 0.0176$, or 1.76 percent (per year)

▶ **Emphasize 1: rate per unit time**

- You earn at the annualized rate of 1.76 percent for 3 months.

(No different from saying you drive at the rate of 55 miles per hour for 5 minutes)

▶ **Emphasize 2: interest rates in percent**

- It is very important when reading, writing, etc., to be clear about whether or not your rates are in percent.
- Rates in percent are simply 100 times the rate stated as a ratio or proportion

E.g., 0.075 or 7.5%

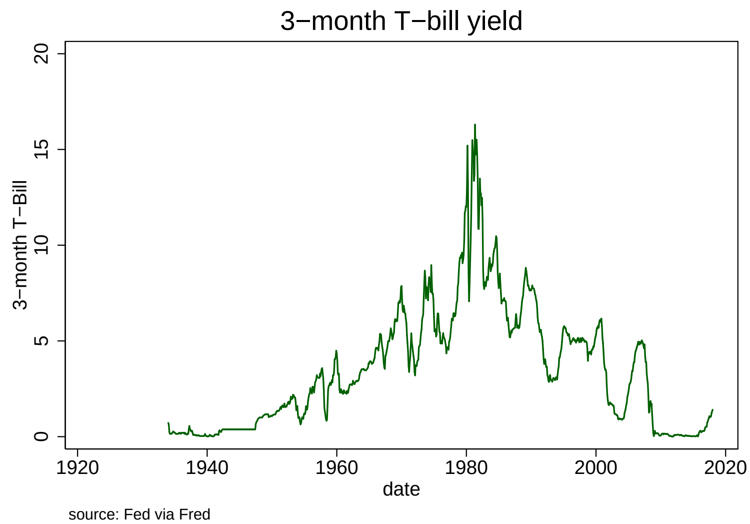
- In formulas in this class, i will be in proportional terms

▶ **4 features of assets/interest rates**

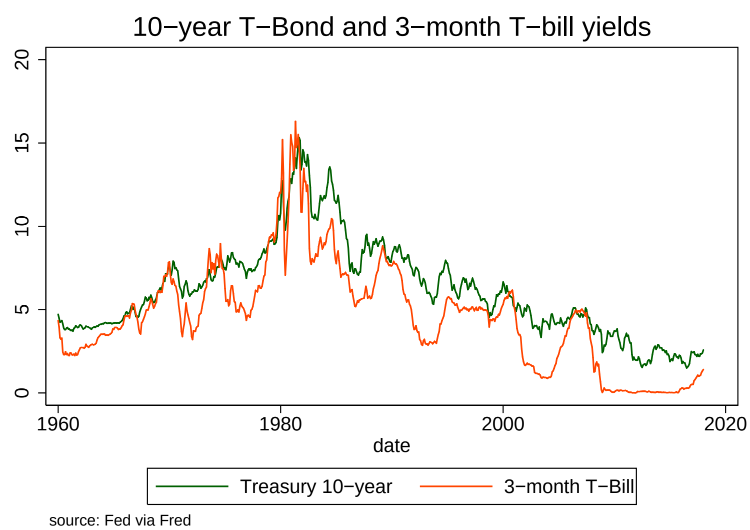
- Interest rates or yields on different assets vary for many reasons, but we lump these into 4 main categories
- maturity or horizon
- risk (of various varieties)
- liquidity
- tax treatment

Let's illustrate these with real data...

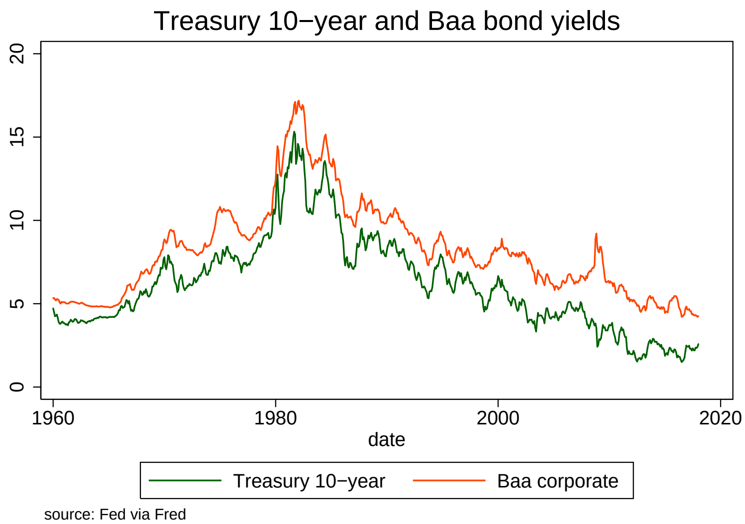
▶ **Start with simple 3-month bill**



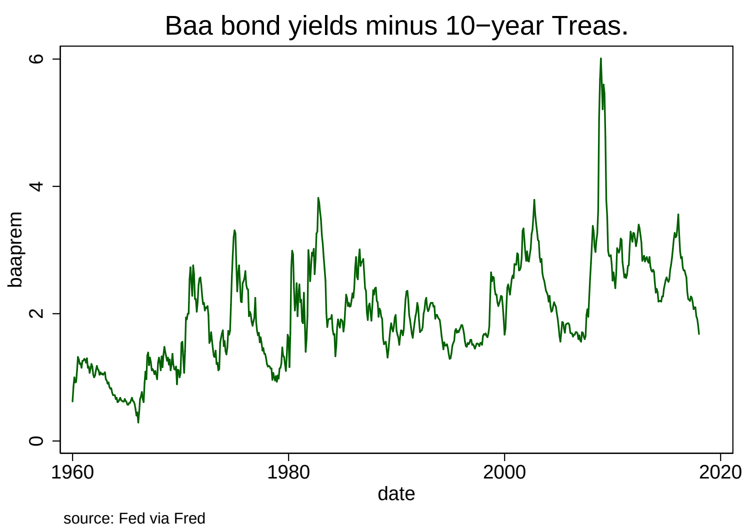
- **Different features**
  - Now we'll present pairs of yields that differ mainly in one or more of these features
- **These differ mainly in maturity**



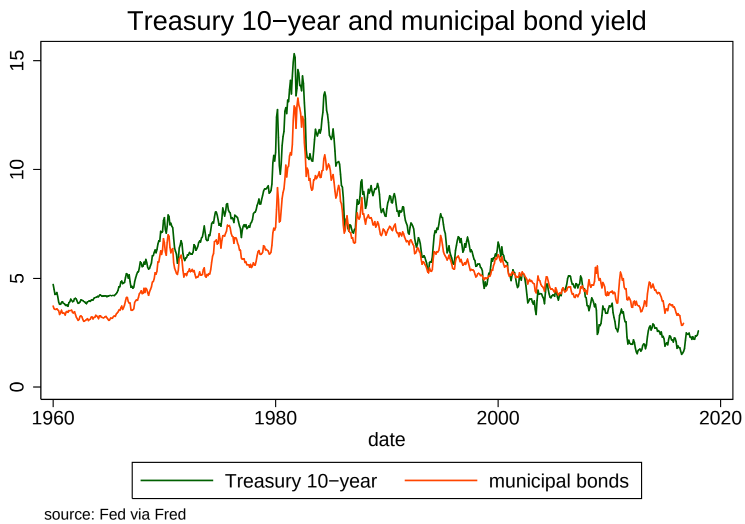
- **These differ mainly in default risk**



- **Premium in Baa bonds**



- **Differ in tax treatment and default risk**



- **Liquidity**
  - And liquidity of the markets is ranked from higher to lower as T Bills, T-Bonds, then corporates and munis

    where the ranking of the latter two can vary by particulars (which corporation, which municipality, etc.)

- **What do you need to know**
  - Basic 'standard' levels of interest rates
  - Basics of what interest rates have done historically

    You should know the reason for any major features in the pictures just given

  - How risk, horizon, and taxes affect yields.

    We have pretty solid analyses of each of these three

  - As for liquidity, we know in general how it affects yields

    But as I've emphasized, it is a more ephemeral thing that we have trouble nailing down.

- **Collections of payments/cash flows/streams of payments**

- **Cash flows**
  - Many financial deals involve promised payments at two points in time

    today and one point in the future

  - Some real-world contracts are like this

A simple loan, a Treasury bill

- But most are not; most involve collection of promised payments at different points in time—a cash flow

  I also call this a stream of payments

▶ **Conventional flows/streams traded in markets**

- 30-year fixed rate mortgage

  Lender gives P to borrower today; in return borrower pays cf every month for 360 months.

- An m-year annuity

  I pay you P today; you pay me cf each year for m years

▶ **Standard cash flows**

- Plain vanilla 10-year coupon bond

  You give me P today, I pay you c once each year until the 10^{th} year and $F + c$ in year 10

- Terminology: c is the coupon value, F is the face (or par) value, P is the price

▶ **How many coupon payments per year**

- In reality, conventions differ a bit on how many times during the year a bond pays a coupon.
- Twice per year is the norm in the U.S.

▶ **Aside:: But in our examples**

- In our examples, we'll tend to assume coupon payments are one time per year
- And that stocks pay dividends once per year
- This makes the formulas neater, but they work the same way with more payments.

▶ **More contracts promising streams**

- m-year discount bond (also called a zero coupon bond)

  You give me P today, I give you F in m years

- Consol or perpetuity

  You pay me P today and I pay you cf once a year forever

▶ **Present value of a set of cash flows is the sum of the present values of the individual payments (giving rise to DCF or discounted cash flow analysis)**

- **Present value of a stream of payments**
  - 1. compute the present value of each individual future payment.
  - 2. Sum them up for the overall present value

- **Plain vanilla M-year coupon bond**
  - Bond pays c each year for M years, and also pays F in the M^{th} year.
  - Present value

$$PV = \left(\sum_{j=1}^{M} \frac{c}{(1+i)^j} \right) + \frac{F}{(1+i)^M}$$

this formula is written with a constant i for discounting every horizon, which we'll later generalize.

- **Internal rate of return and present value**

- **Internal rate of return (IRR)**
  - Suppose some I tell you that an asset pays certain cash flows and I tell you the current price of that asset in financial markets
  - The current price and cash flows imply an IRR: the constant interest rate that makes the present value of the cash flows equal the market price

- **Mechanically**
  - Suppose the stream pays $cf_1, \ldots, cf_M$.
  - Take the present value formula

$$PV = \sum_{j=1}^{M} cf_j/(1+i)^j$$

  - Put the market price, P, in for the PV:

$$P = \sum_{j=1}^{M} cf_j/(1+i)^j$$

  - Solve for i to get the internal rate of return

- **More complete definition**
  - Internal rate of return is the fixed interest rate that makes the *net* present value of an asset (or of investing in a project) equal to zero

- For a generic cash flow, $cf_1, cf_2, \ldots, cf_M$, IRR is the i that sets:

$$0 = \left(\sum_{j=1}^{M} cf_{t+j}/(1+i)^j \right) - P_t$$

▶ **Understanding IRR**

- Any given price for a cash flow will imply a different IRR
- A higher price for a given stream will imply a . . . lower IRR

If you pay more today for the same future payments, the return must be smaller

▶ **Solving for IRR**

- The IRR is the i that solves,

$$P = \sum_{j=1}^{M} cf_j/(1+i)^j$$

- This is an M^{th} order polynomial in i, and we don't have an exact expression for the solution except for special cases (mainly low values of M)
- Even though we do not have an explicit formula for the solution, computers and calculators can easily solve this.

▶ **Terminology note**

- For a bond, the internal rate of return is called the 'yield to maturity' of the bond
- If a bond is selling for P_t, the yield to maturity is the i that solves:

$$P_t = \sum_{j=1}^{M} c/(1+i)^j + F/(1+i)^M$$

- Higher price implies lower yield-to-maturity.
- Higher yield-to-maturity implies lower price

▶ **Remember**

- Prices and yields move in opposite directions

▶ **Yield to maturity vs. holding period return**

▶ **Aside:: 'return' . . . 'holding period return'**

- Where the book says 'return' I'll usually say 'holding period return'

- I'm emphasizing that we are measuring the return you would actually get for owning the asset for a certain period.

  buying it and later selling it.

▶ **Yield to maturity vs. holding period yield**

- Suppose you buy a 10-year bond, and sell it 1 year later.

  What will your return be?

- Hint: it won't be the yield to maturity because you did not hold the bond to maturity.

▶ **Let's start from the basics**

- Yields and returns are always some variation on THE formula
- In this case, we want this version of THE formula:

$$(1 + i) = (FV/PV)^{(1/h)}$$

▶ **Holding period return**

- We are holding the bond for 1 year, so $h = 1$.
- Suppose we buy the bond today for P_t.

  this is the PV

▶ **What is the FV?**

- In a year, what will my proceeds or future value be?

  I get the annual coupon payment, c, and the proceeds from the sale of the bond, P_{t+1}

- Thus,

$$1 + i = \frac{P_{t+1} + c}{P_t}$$

▶ **Re-group this a bit**

$$\begin{aligned} 1 + i &= \frac{P_{t+1} + c}{P_t} \\ i &= \frac{P_{t+1} + c}{P_t} - 1 \\ i &= \frac{P_{t+1} - P_t}{P_t} + \frac{c}{P_t} \end{aligned}$$

▶ **Holding period return**

- Repeat:

$$i = \frac{P_{t+1} - P_t}{P_t} + \frac{c}{P_t}$$

- The two terms are the rate of capital gain (or loss) and the current yield.

  (current yield is defined as coupon divided by current price.)

▶ **Aside:: Capital gain or loss**

- When you hold an asset, the capital gain or loss is any rise or fall in the value (price) over time.
- The rate of capital gain or loss, then, is

$$RCG \equiv \frac{\text{change in price}}{\text{initial price}}$$

▶ **Consol or perpetuity**

- A perpetuity (or 'consol') promises to pay c each period forever.
- I want to go through this to show how some fairly simple math gives useful results
- At a constant interest rate i, the present value of this stream is,

$$\begin{aligned} PV &= c/(1+i) + c/(1+i)^2 + \ldots \\ &= \sum_{j=1}^{\infty} c/(1+i)^j \\ &= c\sum_{j=1}^{\infty} 1/(1+i)^j \end{aligned}$$

- And if I tell you PV and c, the IRR is the i that solves this formula.

▶ **Aside:: Infinite sums**

- You won't have to reproduce this, but for completeness...
- Generally,

$$\sum_{j=1}^{\infty} \rho^j = \ldots$$

$$\rho/(1-\rho)$$

  so long as $|\rho| < 1$

- And more generally,

$$\sum_{j=k}^{\ell} \rho^j = \frac{\rho^k - \rho^{\ell+1}}{1-\rho}$$

- **Using this formula, we get for the consol,**
  - $$PV = c/i$$

    or

    $$i = c/PV$$

  - The IRR of the consol is just the coupon yield.
- **Real vs. nominal interest rates**
- **Nominal intetrest rate**
  - When we say 'interest rate' we usually mean the nominal interest rate denominated in some particular currency

    in this class, U.S. dollars unless otherwise specified

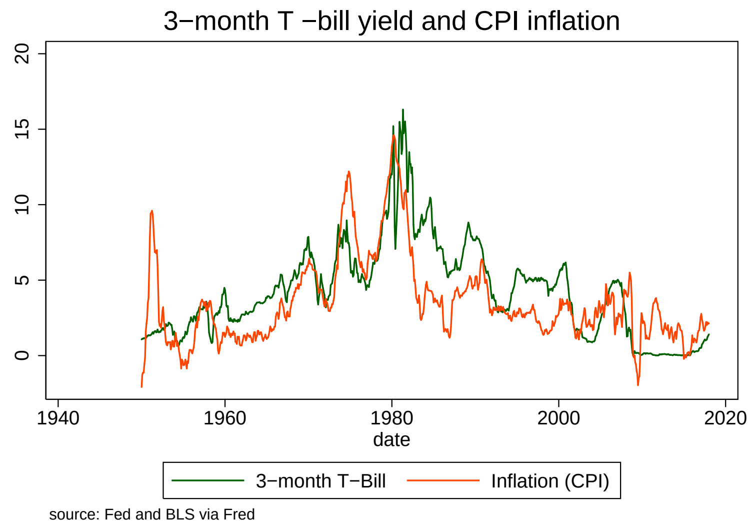
  - The nominal interest rate tells you how fast your value grows in money (that is, dollar) terms

    or yen, or euros, or drachma depending on the denomination of the instrument in question

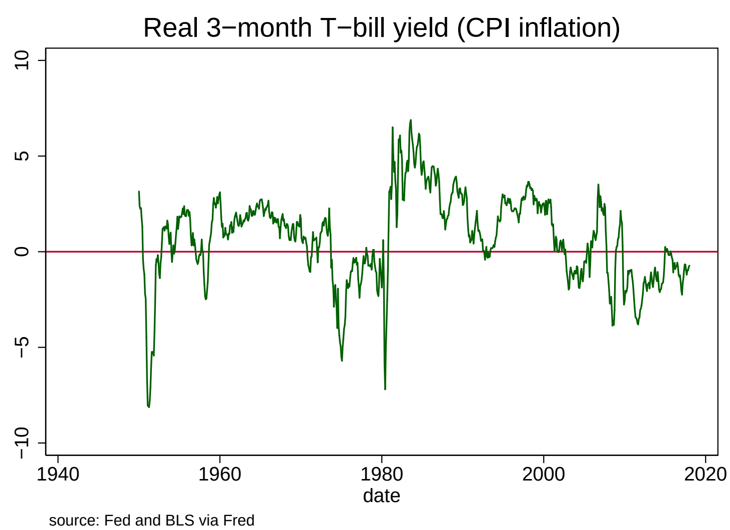
  - The real interest rate is how fast your value is growing in terms of some real thing: e.g., bushels of wheat or in some basket of goods
  - You probably remember, real interest rate equals nominal interest rate minus inflation rate
  - This is called the Fisher relation

    also called the Fisher equation

- **Aside:: Uncertainty**
  - So far we are pretty much ignoring uncertainty
  - If we were acknowledging uncertainty, we'd say 'minus expected inflation' instead of just 'minus inflation'
- **Inflation vs. nominal yield**



- **From the text:**



- Over this period this real interest rate varied between -5 percent and 5 percent.
- In the figure, we proxied expected inflation with the actual inflation rate.

  this is a good proxy over very short horizons like 3-months.

- **Aside:: 'Deriving' the Fisher equation**
  - It is important to understand that going from a return in dollars to a return in yen is just a change in units.
  - And going from a return in dollars to a return in some real item is also just a change in units

▶ **Aside::**

- Both of these are analogous to going from a distance in inches to a distance in centimeters.

▶ **Aside::**

- If you have distance measured in inches and you want units measured in measured in centimeters, you multiply by a factor with units centimeters per inch (≈ 2.54)

▶ **Units on prices**

- Q: As we usually state them, what are the units on prices?
- For concreteness...

▶ **Taco Bell Burrito Supremes (TBBSs)**

▶ **Units on prices**

- The units are dollars per burrito.

  4.49 dollars per burrito at present.

▶ **Prices in real units**

- Consider the price of a new car at say $\$20,000$.
- I want to know the real price in terms of burrito supremes?
- $$20,000 \frac{\text{dollars}}{\text{car}} \times \frac{1}{4.49} \frac{\text{burritos}}{\text{dollar}}$$

- Or, the car costs about 4,454 burritos.

  I would have to forgo 4,454 burritos to buy the car.

▶ **Real rates: converting the units**

- Suppose we have a 1-year zero coupon bond selling for $\$PV$ with a par value of $\$FV$.
- The nominal interest rate is
$$1 + i = \frac{\$FV}{\$PV}$$
- Both FV and PV have units dollars. We'll have to convert both to burritos to get the real rate:
$$1 + i = \frac{FV\text{burritos}}{PV\text{burritos}}$$
- Both FV and PV have units dollars, so I need to multiply each by a conversion factor with units $\frac{\text{burritos}}{\text{dollar}}$
- But PV is in units of dollars today and FV is in units of dollars a year from now.
- We need one conversion for today and one for a year from now.
- Call the price of the burrito today, B_0 dollars per burrito.

  So $1/B_0$ has units burritos per dollar.

- Call the price of a burrito in a year B_1.

▶ **Now just do the conversion**

- $$(1 + r) = \frac{FV \times (1/B_1)}{PV \times (1/B_0)}$$
- But we can simplify a bit

▶ **Real rate**

$$\begin{aligned}
(1 + r) &= \frac{FV \times (1/B_1)}{PV \times (1/B_0)} \\
(1 + r) &= \frac{FV \times B_0}{PV \times B_1} \\
(1 + r) &= \frac{FV}{PV} \times \frac{B_0}{B_1}
\end{aligned}$$

- Or

$$\begin{aligned}
(1 + r) &= \frac{FV}{PV} \times \frac{B_0}{B_1} \\
(1 + r) &= \frac{1 + i}{1 + \pi}
\end{aligned}$$

where π is the rate of inflation in the price of burritos

- **Aside:: inflation rates**
  - The annualized rate of inflation in a price is computed just like a yield:

$$(1+\pi) = \left(\frac{futureprice}{presentprice}\right)^{1/h}$$

where h is how many years pass between the present and future prices.

- **How does our change-of-units result relate to the Fisher equation?**

- **Fisher equation**
  - The Fisher equation is

$$r = i - \pi$$

real rate is nominal rate minus (expected) inflation rate

  - How this relate to:

$$1 + r = \frac{1+i}{1+\pi}$$

  - Take natural logs of both sides of our expression:

$$\ln(1+r) = \ln(1+i) - \ln(1+\pi)$$

  - Now use the approximation $\ln(1+z) \approx z$ for small z:

$$r \approx i - \pi$$

voila, the Fisher equation

- **Aside:: A useful approximation**
  - For small values of z,

$$\ln(1+z) \approx z$$

  - For those with basic calculus, this is a first-order Taylor series approximation.

- **Aside::**

z	$\ln(1+z)$
0.03	0.0295
0.05	0.049
0.10	0.095
0.20	0.182
1×10^6	14

- Pretty good for small z
- At $z = .20$ (twenty percent) the approx. error is getting appreciable, however.

▶ **Aside:: Rocket science!?!**

- Many formulae in finance are greatly simplified using this approximation
- More generally, we'll see that first, second, and third order Taylor series approximations are widely used

    And if you've hear finance folks talking of delta and gamma, they are talking about higher order terms in Taylor series approximations.

▶ **Back to real interest rates**

- For any good G, the real interest rate in units of G is:

$$r_G \approx i - \pi_G$$

  where i is the rate of return in dollars and π_G is the inflation rate in the dollar price of good G.

▶ **Currency conversions**

- Similarly, the nominal return in any currency is, e.g.

$$i_{£} = i_{\$} - \pi_{\$,£}$$

  where $\pi_{\$,£}$ is the rate of change in the exchange rate between dollars and pounds (units $\$/£$)

▶ **What it means: real interest rate**

▶ **The 1-year real interest rate in burrito units**

- The real interest rate in burrito units let's us answer this question:

    If I give up 1 burrito today in order to invest in the bond, how many burritos would I be able to buy in one year when the bond matures.

- The answer is $(1 + r)$

    asuming r is the real interest rate in burrito units

- Financial economics is about how folks make decisions between, say, eating 1 burrito today or 1.03 burritos in a year's time

▶ **Typical real rate units**

- We seldom hear about the real rate in burrito units
- What 'good' is the real rate usually stated in terms of?

  We usually use inflation not in one good, but in terms of a bundle of goods.

- For example, the CPI measures the change in the price of a standard basket of consumer goods
- The personal consumption expenditures inflation index is a different broad measure of inflation

  Based on the conumer basket consumers actually purchased in any given year

- Note: This latter is the Fed's favorite inflation meausure
- Any annualized real rate of interest, r, stated in terms of a broad price index answers the question if I give up one market basket of goods today and invest the money at the real rate r, how many baskets can I buy in 1 year's time.

▶ **What did we cover?**

- Percent changes
- Annualized interest rates
- Present value, including present value of bundles of payments
- Real interest rates.

▶ **But**

- We generally took the interest rate for every horizon to be the same.
- Next time we discuss the term structure of interest rates and the yield curve.

  How do market rates of return vary with the horizon or maturity of the promised payment?