Portfolio theory/Risk management
266: Financial Markets and Institutions

Jon Faust

http://e105.org/e266

April 8, 2017

- ...
  - Today, we'll go from our probability and stats. to portfolio theory.
  - 1. There are many measures of variability of an asset return.

    Just which measure is most relavant must be determined in context.
  - In many discussions we focus on variance (and its relatives std. dev. and correlation, covariance) as our measure of variability.
  - For most of the lecture today, I will assume that the relevant notions of variability are variance and its relatives.

    We'll return to other measures a bit at the end
  - 2. Relevant risk of any given asset return is not first and foremost about variance of that individual asset return.
  - The relevant notion of risk is the covariance of the asset return with the value of other things I own.
  - Covariance will be a main measure of how any given asset contributes to my overall portfolio variance.
  - In choosing an porfolio in the standard case, we look for the right balance of expected return and risk.

    I am willing to accept more risk, but only if it raises my expected return sufficiently.
  - 3. Expected return on a portfolio is a weighted average of the asset expected returns where the weights are the portfolio weights

    By the 'portfolio weight on asset i' we mean the value share of asset i in the overall portfolio.

- 4. The variance of the portfolio is less than the weighted average of the individual variances.

  This is the key: diversification lowers portfolio variance relative to the typical asset variance WITHOUT lowering the expected return relative to the typical asset expected return.

▶ **Aside:: A bit more mathy**

- Suppose I have a bunch of assets which all
  - have the same expected return, r^e
  - have the same variance, v
  - each has zero covariance with each other.

▶ **Aside::**

- I make an equal weighted portfolio of N of these assets.
- The expected return on the portfolio is r^e
- The variance of the portfolio return is v/N

  putting your eggs in 2 as opposed to 1 basket cuts risk in half

- So how many baskets is optimal in this case?

  An infinity of baskets—or in practice, put a tiny amount in each of a large number of assets.

▶ **Portfoilo theory**

- Portfolio theory is trickier when the assets returns have nonzero covariance.
- Like in any microeconomic problem where we are picking a basket of items, we reason in terms of marginal shifts between one thing an another.

▶ **Perfect markets again...**

- The standard perfect competition theory shows how (under certain assumptions) completely independent actions of individual producers and consumers drive equilibrium prices to an efficient place
- And every individual consumer and producer doing what is best for him or her pushes prices to this place.
- A key relation summaries 'good' prices is that the relative prices of goods i and j equal marginal rates of substitution equal marginal rates of transformation

$$P_i/P_j = MRS_{i,j} = MRT_{i,j}$$

- From an individual consumer's standpoint, if I shift \$1 from buying good i to good j, I give up some utility from consuming i and gain some from consuming j.

- When I hit the 'sweet spot' of the best consumption bundle, the gains and losses just balance so that I have no incentive to shift money in either direction.
- Similarly from the production standpoint, the MRT tells us how much j I could produce by shifting inputs from producing i to producing j.

  When the economy is using inputs optimally, nobody can make more money by making any such shift.
- If you want to consider yourself solidly grounded in economics, you deeply understand this

  but this isn't micro class, so I won't go a lot further

▶ **Asset markets**

- In simple portfolio theory we assume every 'product' is just a bundle of two properties: return and risk.
- We reason about moving a marginal dollar from asset i to j.
- When we do this, we gain a bit of the expected return of j and give up a bit of the expected return of i.
- But we also face more of j's risk and less of i's risk
- Thus, the optimal portfolio choice must satisfy some relation like:

$$\frac{r_i^e}{r_j^e} = \frac{\text{risk}_i}{\text{risk}_j}$$

  Where by 'risk$_j$' we mean marginal contribution of asset j to overall portfolio risk.
- To put some meat on these bones, we need some additional assumptions
- The simplest theory is the capital asset pricing model (CAPM)
- Despite it's simplicity, an extremely important theory that drives lots of practical thinking about portfolio choice

▶ **Aside:: Note:**

- Among other things, the assumptions of the CAPM are sufficient so that variability as summarized by the variance family of concepts is everything I care about.
- Thus, under the assumptions of the CAPM we are not leaving anything out in only considering variance and not other aspects of variability.

▶ **CAPM: bottom line**

- If everyone behaves according to the CAPM (and the CAPM assumptions hold) then prices will be driven to the point that for every asset, i, in the economy

$$r_i^e = r_f + \beta_i(r_m^e - r_f)$$

- Note: we can always equivalently talk about prices or returns. Often it is most convenient to talk in terms of returns.
- In this equation, r_f is the risk free rate and r_m^e is the expected return on something called 'the market portfolio'

  basically a value-weighted portfolio of all assets under consideration.

- β_i is a constant describing the marginal contribution of asset i to overall portfolio risk.

▶ **Aside:: Notation vs. book**

- Where the portfolio Appendix writes $E[R_i]$, I write r_i^e.

  Just seems easier to read to me. Hope it doesn't cause confusion.

▶ **Aside::**

- Also, there is a typo in the web appendix
- The main equation in the CAPM is equation (7) in the web appendix.
- But that equation has a typo: the first '+' should be an equal sign.

▶ **Relation to standard micro**

- Let's relate the CAPM equation to the standard relative price equals MRS relation.
- Move r_f to the left side

$$r_j^e - r_f = \beta_j(r_m^e - r_f)$$

- And now take the ratio of this equation for assets i and j

$$\frac{r_i^e - r_f}{r_j^e - r_f} = \frac{\beta_i}{\beta_j}$$

- Relative expected return premia over the risk free rate must equal the relative βs.
- β_j is the contribution to overall portfolio risk of a marginal unit of j
- It turns out that (we won't derive this in this class):

$$\frac{\beta_i}{\beta_j} = \frac{\sigma_{im}}{\sigma_{jm}}$$

where σ_{im} is the covariance of the return of i with the market portfolio return.

- The relative contribution to portfolio risk is given by the relative covariances with the market portfolio.

▶ **Tie into last lecture**

- We said last time that the relevant sense of risk is not 'variance' but covariance with the value of other stuff I own.
- Under the assumptions of the CAPM, the totality of my financial concerns is captured by the return on the market portfolio, thus covariance with the market portfolio is the relevant notion of risk.

▶ **And what is β?**

- In this class, we'll simply say that β_i is (directly) proportional to σ_{im}
- That is, β is a number that has the same sign (pos. or neg.) as σ_{im} and bigger σ_{im} means bigger β.

▶ **One more perpsective on this**

- Suppose we have assets numbered $i = 1, \ldots, n$.
- And we put the share of funds x_i in asset i.

  That is, x_i is our portfolio weight on asset i.

- Expected return:

$$r_P^e = \sum_{i=1}^{n} x_i r_i^e$$

- In words, the portfolio expected return is a weighted average of the underlying asset expected returns using the portfolio weights.
- We know the portfolio variance is not just a weighted average of the variances,
- But there is a way to write the portfolio variance as a simple weighted average.

$$\sigma_P^2 = \sum_{i=1}^{n} x_i \sigma_{iP}$$

  Where σ_{iP} is the covariance of asset i's return with the porfolio return

- Each asset contributes to portfolio variance according to its weight and its *covariance* with the portfolio.

▶ **Thus,**

- Positive covariance of asset i with the portfolio means shifting toward asset i raises overall portfolio variance
- Negative covariance means shifting toward asset i lowers overall portfolio variance

▶ **Let's go back to considering a single asset's return**

▶ **The CAPM equation for each asset return**

- $$r_i^e = r_f + \beta_i(r_m^e - r_f)$$
- An asset that has no covariance with the market portfolio pays the risk free rate on average.
- An asset that covaries positively with the market must earns more than the risk free rate on average

  You must be compensated for the contribution to overall risk

- An asset that covaries negatively with the market earns less than the risk free rate on average.

  You can view the expected return hit you take in holding this asset as a sort of insurance premium just like the premiums you pay on auto insurance.

▶ **Finally, portfolio weights**

- Under the CAPM, that is when risk and return are related as described in the CAPM,. . .
- Everyone optimally holds risky assets in the same proportion: this is called the market portfolio

  Portfolio weights in the risky part of the portfolio are identical for all people

- Thus, everyone holds assets in proportion to their value weight in the overall market
- The only decision individuals make is how much wealth to put in the risk free asset vs. the market portfolio.
- This is a special result that does not hold in more general theories

▶ **Our standard problematic footnote: liquidity**

▶ **Liquidity**

- Liquidity tends to be a separate topic that we have trouble putting in our theoris.
- Thus, we give our best rigorous theory for asset as return
- And then we say, there will be some liquidity premium, ℓ
- We did this when talking about the term structure, and we'll do it here too.

- Thus, for asset i,

$$r^e_{i,t} = r_{f,t} + \beta_i(r^e_{m,t} - r_{f,t}) + l_{i,t}$$

- And that ℓ, as usual, we don't have a great theory for.
- There are zillions of assets out there
- Their returns are somewhat related

▶ **Moving on from variance**

- Much theoretical and real world thinking about risk focusses on variance as the main summary.
- But in reality, it is only under special assumptions that all we need to think about is variance
- Two assets that have the same variance may have very different extreme upside and or downside risk.
- Sophisticated firms take account of much more than just variance and covariance using sophisticated risk models.

▶ **Deeper dive on risk models**

▶ **Risk models**

- And the portfolio theory we just described has an underlying risk model explaining expected return and covariances.

▶ **Review: Risk model**

- A risk model is a large table.

  (Usually hidden within a black box called a computer)

- There is a column for each asset
- Each row gives the returns on each asset in a particular scenario or outcome

  along with the probability of that scenario.

▶ **Risk model for 4 assets**

The joint distribution of 4 returns (in percent)

pr	r_f	r_1	r_2	r_3
1/3	0.5	5	-2	12
1/6	0.5	0	1.75	0
1/3	0.5	17	2.5	0
1/6	0.5	0	-1.5	0

The subscript f stands for risk 'f'ree.

- Suppose we make a portfolio by putting a quarter of wealth in each asset

  So each x weight is 1/4

- Add a column giving the portfolio return in each realization

  The individual returns weighted by 1/4.

The joint distribution of 4 returns (in percent)

pr	r_f	r_1	r_2	r_3	r_P
1/3	0.5	5	-2	12	3.88
1/6	0.5	0	1.75	0	0.56
1/3	0.5	17	2.5	0	5.00
1/6	0.5	0	-1.5	0	-0.25

- Now we can compte the expected return and variance for the portfolio column in the usual way

  based on the outcomes and their probabilities

The joint distribution of 4 returns (in percent)

pr	r_f	r_1	r_2	r_3	r_P
1/3	0.5	5	-2	12	3.88
1/6	0.5	0	1.75	0	0.56
1/3	0.5	17	2.5	0	5.00
1/6	0.5	0	-1.5	0	-0.25
mean					3.01
var.					4.34

▶ **How do we use risk models?**

- We use risk models to assess the mean and variance of different portfolios.
- This can help us to choose among portfolios and to manage (prepare for) risks.
- But we can go beyond expected return and variance

▶ **Richer Example**

- Suppose I have three assets, A, B, C
- My risk model that gives me the the change in value of each asset on any given day under 100 different scenarios
- Values in millions of dollars
- I'll just show the first 10 outcomes as an example

▶ **The first 10 outcomes**

The first 10 outcomes

outcome	pr	a	b	c
1.00	0.02	2.45	2.26	-2.16
2.00	0.01	1.71	2.42	-0.87
3.00	0.00	-1.87	-2.68	2.87
4.00	0.01	0.74	-3.51	3.39
5.00	0.01	-0.15	4.75	-0.28
6.00	0.01	0.69	-1.40	3.20
7.00	0.00	-1.64	-2.45	-1.06
8.00	0.02	1.16	0.37	-1.23
9.00	0.01	2.69	1.67	-2.51
10.00	0.00	-3.76	-0.79	1.00

▶ **Add the overall net outcome**

The net result in the 10 outcomes

outcome	pr	a	b	c	net
1.00	0.02	2.45	2.26	-2.16	2.54
2.00	0.01	1.71	2.42	-0.87	3.26
3.00	0.00	-1.87	-2.68	2.87	-1.68
4.00	0.01	0.74	-3.51	3.39	0.62
5.00	0.01	-0.15	4.75	-0.28	4.33
6.00	0.01	0.69	-1.40	3.20	2.49
7.00	0.00	-1.64	-2.45	-1.06	-5.15
8.00	0.02	1.16	0.37	-1.23	0.30
9.00	0.01	2.69	1.67	-2.51	1.85
10.00	0.00	-3.76	-0.79	1.00	-3.55

▶ **Messy**

- In reality, there might be values for hundreds of assets and liabilities in thousands of scenarios.
- Thus, we have some standard ways of summarizing the risk as represented in the model.

▶ **Value at Risk, VaR**

- One way to summarize risk is known as value at risk.

  often written VaR

▶ **Value at risk**

- Pick a time horizon.

say 1 day.

- Pick a large probability

  e.g. 95%

- 1-day, 95% VaR defined as the dollar value, $\$V$, such that you have only a 5% chance of losing more than $\$V$ in 1 day.

  So 95% of the time you lose no more than $\$V$.

▶ **Computing VaR**

- In the table, sort from worst to best net outcomes
- Go down the rows until you have a cumulative probability of 5 percent
- And read off the size of the loss in the event on that row

  this is your 1-day 95% VaR

▶ **Example**

- sort worst to best
- then read off value such that you lose this much or more only 5% of the time

▶ **Sorted outcome**

Sorted net outcomes, worst to best
(cumpr is cumulation of pr)

outcome	pr	cumpr	a	b	c	net
20.00	0.00	0.00	-0.98	-5.02	-2.50	-8.49
10.00	0.01	0.02	-0.22	1.18	-6.88	-5.93
74.00	0.01	0.03	-4.00	-2.64	1.85	-4.79
85.00	0.01	0.03	-1.29	-2.08	-1.31	-4.67
33.00	0.01	0.04	0.46	-4.13	-0.73	-4.41
75.00	0.01	0.05	0.83	-2.53	-2.52	-4.21
79.00	0.01	0.06	-2.77	-2.18	0.82	-4.13
60.00	0.00	0.07	-2.55	-1.45	0.23	-3.77
68.00	0.00	0.07	-1.35	-4.21	1.82	-3.75
97.00	0.02	0.09	0.52	-0.35	-3.58	-3.41

▶ **VaR**

- My 1-day, 95% VaR is -4.21 million dollars

  The model says that I'll do this bad or worse 5% of the time.

▶ **VaR**

- Two portfolios with the same variance could have very different features in other regards

- The 1-day 95% VaR is one way the two might vary.
- And big losses may be of particular interest to decisionmakers

  They don't want to go broke.

- Thus VaR is one way financial firms condense their risk model into a few numbers for the purpose of decisionmaking.

► **You'll see on the problem set how JPM Chase uses VaR in risk management.**